DOMAINS OF COMMUTATIVE C*-SUBALGEBRAS

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Abstract. A C*-algebra is determined to a great extent by the partial order of its commutative C*-algebras. We study order-theoretic properties of this dcpo. Many properties coincide: the dcpo is, equivalently, algebraic, continuous, meet-continuous, atomistic, quasi-algebraic, or quasi-continuous, if and only if the C*-algebra is scattered. For C*-algebras with enough projections, these properties are equivalent to finite-dimensionality. Approximately finite-dimensional elements of the dcpo correspond to Boolean subalgebras of the projections of the C*-algebra. Scattered C*-algebras are finite-dimensional if and only if their dcpo is Lawson-scattered. General C*-algebras are finite-dimensional if and only if their dcpo is order-scattered.

Introduction

C*-algebras play a role in many areas. They arose from quantum mechanics, but can be created in canonical ways from many other mathematical objects. Time and again, passing from the input object to the resulting C*-algebra retains many salient features. In this way C*-algebras have advanced the study of graphs [Rae05], minimization of automata [BKP12], semantics of linear logic [DR95], geometry of interaction [Gir11], bisimulation of labelled Markov processes [MOPW04, SD80, Koz81, MP09, LS91], and operational semantics of probabilistic languages [DHW03, DW06], as well as quantum information theory [Key02].

A recently successful programme is to study a C*-algebra A through the partial order $\mathcal{C}(A)$ of its commutative C*-subalgebras. This partial order determines the C*-algebra to a great extent [Ham11, DH15, HR14], and has therefore become popular as a substitute for the C*-algebra itself [Heu14b, Spi12, Lin15, Rey12, BH14, Rey14, BH12, HLS09, Ham15, Heu14c, Heu14a]. The intuition is clearest in the case of quantum theory. There, the C*-algebra models all observations one can possibly perform on a quantum system. However, not all observations may be performed simultaneously, but only those that live together in a commutative C*-subalgebra C. There is an inherent notion of approximation: if $C \subseteq D$, then D contains more observations, and hence provides more information.

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This sort of informational approximation is more commonly studied in *domain theory* [AJ94]. As we are speaking of a continuous amount of observables, but in practice only have access to a discrete number of them, we are most interested in partial orders where every element can be approximated by empirically accessible ones. Domain theory has a variety of notions modelling this intuition. We show that they all coincide in our setting, as the following are equivalent:

- the C*-algebra A is scattered (Section 1);
- the partial order $\mathcal{C}(A)$ is algebraic (Section 2);
- the partial order $\mathcal{C}(A)$ is continuous (Section 3);
- the partial order $\mathcal{C}(A)$ is meet-continuous (Section 4);
- the function $\mathcal{C}(f)$ is Scott-continuous for each injective $*$-homomorphism $f\colon B \to A$;
- the partial order $\mathcal{C}(A)$ is atomistic (Section 5);
- the partial order $\mathcal{C}(A)$ is quasi-algebraic (Section 6);
- the partial order $\mathcal{C}(A)$ is quasi-continuous (Section 6).

This makes precise exactly 'how much approximate finite-dimensionality' on the algebraic side is required for these desirable notions of approximation on the domain-theoretic side. It is satisfying that these notions robustly coincide with the established algebraic notion of scatteredness, which is intimately related to approximate finite-dimensionality.

We also study finite-dimensionality of A in terms of the partial order $\mathcal{C}(A)$:

- a C*-algebra A is finite-dimensional if and only $\mathcal{C}(A)$ is Lawson-scattered (Section 7);
- a C*-algebra A is finite-dimensional if and only if $\mathcal{C}(A)$ is order-scattered (Section 8);

Finally, we study how such approximations respect projections, which form an important part of traditional C*-algebra theory (Section 9):

- a C*-algebra A with enough projections (a so-called AW*-algebras) is finite-dimensional if and only if $\mathcal{C}(A)$ is continuous, if and only if $\mathcal{C}(A)$ is algebraic;
- the partial order $\mathcal{C}_{\mathrm{AF}}(A)$ of commutative approximately finite-dimensional C*-subalgebras of a C*-algebra A is isomorphic to the domain of Boolean subalgebras of the projections of A.

For related but different work, see also [DD03, DDP03, Ren14, Cho14, AV93, Abr05].

1. C*-algebras

For detailed information about C*-algebras, we refer to [Dav91, RK83]; here we briefly recall the ingredients we need. To introduce the idea intuitively, consider *transition systems* with n states. These can be represented as n-by-n matrices with entries 0 or 1. Linking two transitions becomes matrix multiplication (over the Boolean semiring $\{0,1\}$, so with maximum instead of addition), and reversing transitions becomes matrix transpose. *Labelled transition systems* have transition matrices for each action in a whole set of labels. More generally, *probabilistic transition systems* can be represented as matrices with nonnegative real entries. Linking transitions is again matrix multiplication, reversing transitions is matrix transpose, and *labelled Markov systems* have different such probability matrices for transitions between states for a whole set of labels [Pan09]. More generally still, *quantum systems* replace probabilities by complex numbers, that now model the amplitude of one computational state evolving into another [MC00]. (Taking the absolute square of the amplitudes recovers the probabilistic case.) Linking transitions is still matrix multiplication, reversing transitions is conjugate transpose. Again, one can have different transition

matrices for different computation steps. This leads to the algebra of all 2^n-by-2^n complex matrices; the transition matrices together generate a subalgebra. C*-algebras define such operational semantics for situations with possibly infinitely many states rather than $n < \infty$. The star refers to the reversal operation. Here are the definitions.

Definition 1.1. A *norm* on a complex vector space V is a function $\| - \| \colon V \to [0, \infty)$ satisfying

- $\|v\| = 0$ if and only if $v = 0$;
- $\|\lambda v\| = |\lambda| \|v\|$ for $\lambda \in \mathbb{C}$;
- $\|v + w\| \leq \|v\| + \|w\|$.

A *Banach space* is a normed vector space that is complete in the metric $d(v, w) = \|v - w\|$.

Example 1.2. An *inner product* on a vector space V is a map $\langle - \mid - \rangle \colon V \times V \to \mathbb{C}$ that:

- is linear in the second variable;
- is conjugate symmetric: $\langle v \mid w \rangle = \overline{\langle w \mid v \rangle}$;
- satisfies $\langle v \mid v \rangle \geq 0$ with equality only when $v = 0$.

An inner product space V is a *Hilbert space* when the norm $\|v\| = \sqrt{\langle v \mid v \rangle}$ makes it a Banach space. For example, $\mathbb{C}^n$ with its usual inner product is a Hilbert space.

Definition 1.3. A complex vector space A is a (unital) *algebra* when it carries a bilinear associative multiplication $A \times A \to A$ with a unit $1 \in A$ satisfying $1a = a = a1$. It is *commutative* when $ab = ba$ for all $a, b \in A$. A $*$-algebra is an algebra with an anti-linear map $* \colon A \to A$ satisfying:

- $(a^*)^* = a$;
- $(ab)^* = b^* a^*$.

A *C*-algebra* is a $*$-algebra that is simultaneously a Banach space with:

- $\|ab\| \leq \|a\| \|b\|$;
- $\|a^* a\| = \|a\|^2$.

Example 1.4. As mentioned, the set of all n-by-n complex matrices is a C*-algebra, with the star being conjugate transpose. The infinite-dimensional version is this: the space $B(H)$ of all continuous linear maps $a \colon H \to H$ on a Hilbert space H is a C*-algebra as follows. Addition and scalar multiplication are defined componentwise by $a + b \colon v \mapsto a(v) + b(v)$, multiplication is composition by $ab \colon v \to a(b(v))$, and 1 is the identity map $v \mapsto v$. The star of $a \colon H \to H$ is its *adjoint*, *i.e.*, the unique map satisfying $\langle v \mid a(w) \rangle = \langle a^*(v) \mid w \rangle$ for each $v, w \in H$. The norm is given by $\|a\| = \sup\{\|a(v)\| \mid v \in H, \|v\| = 1\}$. Notice that this C*-algebra is noncommutative (unless H is one-dimensional).

The previous example is in fact prototypical, as the following theorem shows. A linear map $f \colon A \to B$ between C*-algebras is a unital $*$-*homomorphism* when $f(ab) = f(a)f(b)$, $f(a^*) = f(a)^*$, and $f(1) = 1$. If f is bijective we call it a $*$-*isomorphism*, and write $A \simeq B$. Every $*$-homomorphism is automatically continuous, and is even an isometry when it is injective [RK83, 4.1.8]. A C*-algebra B is a *C*-subalgebra* of a C*-algebra A when $B \subseteq A$, and the inclusion $B \to A$ is a unital *-homomorphism. Since the inclusion must be an isometry, it follows that every C*-subalgebra of A is a closed subset of A.

Theorem 1.5 (Gelfand–Neumark)**.** [Tak00, 9.18]. *Any C*-algebra is $*$-isomorphic to a C*-subalgebra of $B(H)$ for a Hilbert space H.* □

The above C*-algebra is noncommutative. Here is an example of a commutative one.

Example 1.6. The vector space $\mathbb{C}^n$ is a commutative C*-algebra under pointwise operations. It sits inside the algebra $B(\mathbb{C}^n)$ of n-by-n matrices as the subalgebra of diagonal ones, illustrating Theorem 1.5. The infinite version is as follows. Write $C(X)$ for the set of all continuous functions $f\colon X \to \mathbb{C}$ on a compact Hausdorff space X. It becomes a commutative C*-algebra as follows: addition and scalar multiplication are pointwise, *i.e.*, $f+g\colon x \mapsto f(x)+g(x)$, multiplication is componentwise $fg\colon x \mapsto f(x)g(x)$, the unit is the function $x \mapsto 1$, the star is $f^*\colon x \mapsto \overline{f(x)}$, and the norm is $\|f\| = \sup_{x\in X} |f(x)|$.

The above example is again prototypical for commutative C*-algebras.

Theorem 1.7 (Gelfand duality)**.** [Tak00, 1.3.10, 1.4.4, 1.4.5]. *Any commutative C*-algebra is $*$-isomorphic to $C(X)$ for some compact Hausdorff space X, called its* spectrum. □

The previous theorem extends to a categorical duality, but we only need the following.

Proposition 1.8. *Let A be a commutative C*-algebra with spectrum X. If $X \to Y$ is a continuous surjection onto a compact Hausdorff space Y, then Y is homeomorphic to the spectrum of a C*-subalgebra of A. Conversely, if a C*-subalgebra of A has spectrum Y, there is a continuous surjection $X \to Y$.*

Proof. If $q\colon X \to Y$ is a continuous surjection, then $B = \{f \circ q \mid f \in C(Y)\}$ is a C*-subalgebra of A. Conversely, if B is a C*-subalgebra of A, define a equivalence relation $\sim_B$ on X by setting $x \sim_B y$ if and only if $b(x) = b(y)$ for each $b \in B$. The quotient $Y = X/{\sim_B}$ is a compact Hausdorff space and it follows that $C(Y)$ is *-isomorphic to B. For details, see [Wea01, 5.1.3]. □

Commutative C*-subalgebras. We now come to our main object of study. The usual way of approximating labelled transition systems by simpler ones is to identify some *bisimilar* states, that is, to take a certain quotient of the (topological) space of states [Pan09, DD03]. In the commutative case, this comes down to considering C*-subalgebras by the previous proposition. When generally describing (quantum) systems C*-algebraically, observables become *self-adjoint* elements $a = a^* \in A$. These correspond [Tak00, 4.6] to injective $*$-homomorphisms $C(\sigma(a)) \to A$ via the *spectrum* of a, the compact Hausdorff space

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1_A \text{ is not invertible}\},$$

linking observables to commutative C*-subalgebras. The following definition captures the main structure of approximation on the algebraic side.

Definition 1.9. For a C*-algebra A, define

$$\mathcal{C}(A) = \{C \subseteq A \mid C \text{ is a commutative C*-subalgebra}\},$$

partially ordered by inclusion: $C \leq D$ when $C \subseteq D$.

Let us consider some elementary domain-theoretic properties of $\mathcal{C}(A)$ now. For detailed information about domain theory, we refer to [AJ94, GHK+03]. Let $\mathcal{C}$ be a partially ordered set. We think of its elements as partial computations or observations, and the partial order $C \leq D$ as "D provides more information about the eventual outcome than C". With this interpretation, it is harmless to consider *downsets*, or *principal ideals*, instead of $C \in \mathcal{C}$:

$$\downarrow C = \{D \in \mathcal{C} \mid D \leq C\}.$$

Dually, it is also of interest to consider *upsets*, or *principal filters*, consisting of all possible expansions of the information contained in $C \in \mathcal{C}$:

$$\uparrow C = \{D \in \mathcal{C} \mid D \geq C\}.$$

This extends to subsets $\mathcal{D} \subseteq \mathcal{C}$ as:

$$\downarrow \mathcal{D} = \bigcup_{D \in \mathcal{D}} \downarrow D, \qquad \uparrow \mathcal{D} = \bigcup_{D \in \mathcal{D}} \uparrow D.$$

If $\mathcal{D}$ has a least upper bound in $\mathcal{C}$, it is denoted by $\bigvee \mathcal{D}$. Furthermore, $\mathcal{D}$ is called *directed* if for each $D_1, D_2 \in \mathcal{D}$ there is a $D_3 \in \mathcal{D}$ such that $D_1, D_2 \leq D_3$. This can be interpreted as saying that the partial computations or observations in $\mathcal{D}$ can always be compatibly continued without leaving $\mathcal{D}$. We write $C = \bigvee \mathcal{D}$ when $\mathcal{D}$ is directed and has C as a least upper bound. Similarly, we write $\bigwedge \mathcal{D}$ for a greatest lower bound, when it exists. For two-element sets $\mathcal{D}$ we just write the *meet* $\bigwedge\{D_1, D_2\}$ as $D_1 \wedge D_2$.

Definition 1.10. A partially ordered set $\mathcal{C}$ is a *directed-complete partial order* (dcpo) if each directed subset of $\mathcal{C}$ has a least upper bound.

Proposition 1.11. *If A is any C*-algebra, then $\mathcal{C}(A)$ is a dcpo.*

Proof. The least upper bound $\bigvee D_i$ of a directed subset $\{D_i\} \subseteq \mathcal{C}(A)$ is the closure $\overline{\bigcup D_i}$ of $\bigcup D_i$. In particular, if A is finite-dimensional, then $\bigvee D_i = \bigcup D_i$. See also [Spi12]. □

The assignment $A \mapsto \mathcal{C}(A)$ also extends to functions. A function f between partially ordered sets is *monotone* when $C \leq D$ implies $f(C) \leq f(D)$; an *order isomorphism* is a monotone bijection with a monotone inverse. A unital $*$-homomorphism $f\colon A \to B$ maps (commutative) C*-subalgebras $C \subseteq A$ to (commutative) C*-subalgebras $f[C] \subseteq B$ [RK83, 4.1.9]. Thus $\mathcal{C}$ extends to a functor from the category of C*-algebras and unital $*$-homomorphisms to that of dcpos and monotone functions by $\mathcal{C}(f)\colon C \mapsto f[C]$. The property in the following proposition is also called *Scott continuity.*

Proposition 1.12. *If $f\colon A \to B$ is a unital $*$-homomorphism between C*-algebras, then*

$$\bigvee \mathcal{C}(f)[D_i] = \mathcal{C}(f)\big(\bigvee D_i\big)$$

for any directed subset $\{D_i\} \subseteq \mathcal{C}(A)$.

Proof. Since $\mathcal{C}(f)$ is monotone, $\mathcal{C}(f)[\mathcal{D}]$ is directed. Now

$$f\left[\bigvee D_i\right] = f\left[\overline{\bigcup D_i}\right] \subseteq \overline{f\left[\bigcup D_i\right]} = \overline{\bigcup \{f[D_i]\}},$$

where the inclusion holds because unital $*$-homomorphisms are continuous. Conversely,

$$\begin{aligned} \overline{\bigcup \{f[D_i]\}} = \overline{f\left[\bigcup D_i\right]} &\subseteq \overline{f\left[\overline{\bigcup D_i}\right]} \\ &= \overline{f\left[\bigvee D_i\right]} = f\left[\bigvee D_i\right], \end{aligned}$$

where the last equality holds because C*-subalgebras are closed. Hence

$$\bigvee \{f[D_i]\} = \overline{\bigcup \{f[D_i]\}} = f\left[\bigvee D_i\right],$$

which finishes the proof. □

The dcpo $\mathcal{C}(A)$ is of interest because it determines the C*-algebra A itself to a great extent. The following theorem illustrates this; see also [DH15, Lin15, Heu14b, Ham11].

Theorem 1.13. [Ham11, 3.4]. *Let A, B be commutative C*-algebras. An arbitrary order isomorphism $\psi\colon \mathcal{C}(A) \to \mathcal{C}(B)$ allows a $*$-isomorphism $f\colon A \to B$ with $\mathcal{C}(f) = \psi$ that is unique unless A is two-dimensional.* □

It already follows that for arbitrary C*-algebras A, the partial order on $\mathcal{C}(A)$ determines the C*-algebra structure of each individual element of $\mathcal{C}(A)$. Indeed, if $C \in \mathcal{C}(A)$, then $\downarrow C$ is order isomorphic to $\mathcal{C}(C)$, and since C is a commutative C*-algebra, it follows that the partially ordered set $\downarrow C$ determines the C*-algebra structure of C.

Approximate finite-dimensionality. In practice, within finite time one can only measure or compute up to finite precision, and hence can only work with (sub)systems described by finite-dimensional C*-subalgebras. Therefore one might think that the natural extension is for the finite-dimensional C*-subalgebras to be dense in the whole C*-algebra.

Definition 1.14. A C*-algebra A is called *(locally) approximately finite-dimensional* when for each $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$ there exist a finite-dimensional C*-subalgebra $B \subseteq A$ and $b_1, \ldots, b_n \in B$ such that $\|a_i - b_i\| < \varepsilon$ for any $i = 1, \ldots, n$.

Let us point out that we do not, as some authors do, restrict approximately finite-dimensional C*-algebras to have countable dimension. (See also [FK10].) It turns out that a countably-dimensional C*-algebra A is approximately finite-dimensional precisely when $A = \overline{\bigcup_{n\in\mathbb{N}} D_n}$ for a sequence $D_1 \subseteq D_2 \subseteq \cdots$ of finite-dimensional C*-subalgebras. These can be classified in several ways, for instance by means of Bratteli diagrams [Bra72] or by K-theory [WO93]. The following lemma shows that the notions of approximate finite-dimensionality of [Bra72] and [Kus10] coincide in the commutative case. Namely, a commutative C*-algebra is approximately finite-dimensional if and only if its spectrum is *totally disconnected*: that is, when its connected components are exactly the singletons. Countable-dimensionality gives the additional requirement that the spectrum be second-countable [Bra74, Proposition 3.1].

Lemma 1.15. *The following are equivalent for a compact Hausdorff space X:*

- *$C(X)$ is approximately finite-dimensional;*
- *X is totally disconnected;*
- *$C(X) = \overline{\bigcup D_i}$ for a directed set $\{D_i\}$ of finite-dimensional C*-subalgebras of $C(X)$.*

Proof. Assume $C(X)$ is approximately finite-dimensional and let $x, y \in X$ be distinct points. Urysohn's lemma gives $f \in C(X)$ with $f(x) = 1 \neq 0 = f(y)$. By Definition 1.14, there exist a finite-dimensional C*-subalgebra B and $g \in B$ with $\|f - g\| < \frac{1}{2}$. This implies that $g(x) \neq g(y)$. Since B is finite-dimensional, Lemma 3.3 below makes the set $\{y \in X \mid \forall f \in B\colon f(x) = f(y)\}$ clopen. Hence x and y cannot share a connected component, and X is totally disconnected.

Next, let X be totally disconnected. Distinct points $x, y \in X$ induce a clopen subset $C \subseteq X$ containing x but not y. Hence the characteristic function of C is continuous, and thus a projection. So the projections of $C(X)$ separate X. It follows from the Stone–Weierstrass theorem that the projections span an algebra B that is dense in $C(X)$. Let D_i be the algebras spanned by finitely many projections and 1_X. This clearly gives a directed family that consists solely of finite-dimensional C*-subalgebras and has least upper bound B, so

$\overline{\bigcup D_i} = C(X)$. Finally, if there is a directed set of finite-dimensional C*-subalgebras whose union lies dense in $C(X)$, it follows easily that $C(X)$ is approximately finite-dimensional. $\square$

Example 1.16. Let X be the Cantor set. Then $C(X)$ is an approximately finite-dimensional commutative C*-algebra of countable dimension. Since there exists a continuous surjection $X \to [0,1]$, there is a C*-subalgebra of $C(X)$ that is $*$-isomorphic to $C([0,1])$ by Proposition 1.8. This C*-subalgebra is not approximately finite-dimensional because $[0,1]$ is not totally disconnected.

The following useful lemma explains the terminology 'approximately' by linking the topology of a C*-algebra to approximating C*-subalgebras.

Lemma 1.17. [WO93, Proposition L.2.2] *Let A be a C*-algebra and $\mathcal{D}$ a directed family of C*-subalgebras with $A = \overline{\bigcup \mathcal{D}}$. For each $a \in A$ and $\varepsilon > 0$, there exist $D \in \mathcal{D}$ and $x \in D$ satisfying $\|a - x\| < \varepsilon$. If a is a* projection, *i.e., $a = a^2 = a^*$, then x can be chosen to be a projection as well.* $\square$

It will turn out that approximate finite-dimensionality of A does not correspond to nice order-theoretic properties of $\mathcal{C}(A)$. We will need the following more subtle notion. In general, we will use quite some point-set topology of totally disconnected spaces, as covered *e.g.* in [GHK$^+$03].

Definition 1.18. A topological space is called *scattered* if every nonempty closed subset has an isolated point.

Equivalently, a topological space X is scattered if there is no continuous surjection $X \to [0,1]$ [Sem71, Theorem 8.5.4]. Scattered topological spaces are always totally disconnected, so commutative C*-algebras with scattered spectrum are always approximately finite-dimensional.

Example 1.19. Any discrete topological space is scattered, and any finite discrete space is additionally compact Hausdorff, but there are more interesting examples.

The one-point compactification of the natural numbers is scattered, as well as compact Hausdorff. This is homeomorphic to the subspace $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$ under the usual Euclidean topology.

More generally, any ordinal number α is scattered under the order topology. A basis for this topology is given by the intervals $\{\delta \mid \beta < \delta < \gamma\}$ for ordinals $\beta, \gamma \leq \alpha$. If α is a limit ordinal, then $\alpha + 1$ is furthermore compact Hausdorff [SS70, Counterexample 43].

There is also an established notion of scatteredness in general C*-algebras A, which can be defined as follows [Jen77]. A *positive functional* on A is a continuous linear map $f\colon A \to \mathbb{C}$ satisfying $f(a^*a) \geq 0$. The space of positive functionals of unit norm forms a convex set, whose extremal points are called *pure*. A C*-algebra is called *scattered* when each positive functional can be written as the countable sum of pure positive functionals.

Example 1.20. An operator $f \in B(H)$ on a Hilbert space H is *compact* when it is a limit of operators of finite rank. If H is infinite-dimensional, the compact operators form a proper ideal $K(H) \subseteq B(H)$, and all self-adjoint elements of $K(H)$ have countable spectrum [Con90, VII.7.1]. It follows that the C*-algebra $K(H) + \mathbb{C}1_H$ is scattered [Hur78].

The following theorem connects the notions of approximately finite-dimensional algebras, scattered topological spaces, and scattered C*-algebras.

Theorem 1.21. *The following are equivalent for C*-algebras:*

1. *A is scattered;*
2. *each $C \in \mathcal{C}(A)$ is approximately finite-dimensional;*
3. *each $C \in \mathcal{C}(A)$ has totally disconnected spectrum;*
4. *each maximal $C \in \mathcal{C}(A)$ has scattered spectrum;*
5. *no $C \in \mathcal{C}(A)$ has spectrum $[0, 1]$.*

Proof. The equivalence between (1) and (2) follows from [Kus10, 2.2], and the equivalence between (2) and (3) is proven in Lemma 1.15. Assume that all maximal commutative C*-subalgebras have scattered spectrum. Since by Zorn's lemma every commutative C*-subalgebra is contained in a maximal one, it follows from [FHH$^+$01, 12.24] that all commutative C*-subalgebras have scattered and hence totally disconnected spectra. Conversely, suppose that every commutative C*-subalgebra has totally disconnected spectra and let C be a maximal commutative C*-subalgebra. Since every C*-subalgebra of C has totally disconnected spectrum, there can be no C*-subalgebra of C with spectrum homeomorphic to $[0, 1]$. It now follows that C has scattered spectrum [Sem71, 8.5.4], which establishes the equivalence of (3) and (4). We show that (1) implies (5) by contraposition. Assume that A has a commutative C*-subalgebra with spectrum $[0, 1]$, and write $a\colon [0, 1] \to [0, 1]$ for the self-adjoint element $a(x) = x$ of A. Because $\sigma(a) = [0, 1]$ is uncountable, A cannot be scattered. Finally, we show that (5) implies (4) by contraposition. Assume that some maximal commutative C*-subalgebra M has non-scattered spectrum X. Then there is a continuous surjection $X \to [0, 1]$, and it follows that M, and hence A, has a commutative C*-subalgebra whose spectrum is (homeomorphic to) $[0, 1]$. □

2. Algebraicity

In this section we characterize C*-algebras A for which $\mathcal{C}(A)$ is algebraic. First recall what the latter notion means. Consider elements B, C of a dcpo $\mathcal{C}$. The element C could contain so much information that it is practically unobtainable. What does it mean for B to approximate C empirically? One answer is: whenever C is the eventual observation of increasingly fine-grained experiments D_i, then all information in B is already contained in a single one of the approximants D_i. More precisely: we say that B is *way below* C and write $B \ll C$ if for each directed subset $\{D_i\}$ of $\mathcal{C}$ the inequality $C \leq \bigvee D_i$ implies that $B \leq D_i$ for some i. Define:

$$\twoheaddownarrow C = \{B \in \mathcal{C} \mid B \ll C\}, \qquad \twoheaduparrow C = \{B \in \mathcal{C} \mid C \ll B\}.$$

With this interpretation, C is empirically accessible precisely when $C \ll C$. Such elements are called *compact*, and the subset they form is denoted by $\mathcal{K}(\mathcal{C})$.

Definition 2.1. A dcpo is *algebraic* when each element satisfies $C = \bigvee \big(\mathcal{K}(\mathcal{C}) \cap \twoheaddownarrow C\big)$.

We start by identifying the compact elements of $\mathcal{C}(A)$. If K is a closed subspace of a compact Hausdorff space X, define

$$C_K = \{f \in C(X) \mid f \text{ is constant on } K\},$$

which is clearly a C*-subalgebra of $C(X)$. The following lemma gives a convenient way to construct directed subsets of $\mathcal{C}(A)$.

Lemma 2.2. *Let A be a C*-algebra, and $C \subseteq A$ a commutative C*-subalgebra with spectrum X. If $P \subseteq X$ is finite, then*

$$\{\bigcap_{p\in P} C_{\overline{U_p}} \mid U_p \text{ open neighbourhood of } p\}$$

is a directed family in $\mathcal{C}(A)$ with supremum C.

Proof. If U_p and V_p are open neighbourhoods of p, then so is $U_p \cap V_p$. Moreover

$$\overline{U_p \cap V_p} \subseteq \overline{U_p} \cap \overline{V_p} \subseteq \overline{U_p},$$

and so $C_{\overline{U_p}} \subseteq C_{\overline{U_p \cap V_p}}$. Similarly $C_{\overline{V_p}} \subseteq C_{\overline{U_p \cap V_p}}$. Hence $\bigcap C_{\overline{U_p}}$ and $\bigcap C_{\overline{V_p}}$ are both contained in $\bigcap C_{\overline{U_p \cap V_p}}$, making the family directed.

To show that the supremum is $C(X)$, it suffices to show that the supremum separates all points of X by the Stone–Weierstrass Theorem [RK83, 3.4.14]. First, the supremum clearly contains all constant functions, since $x \mapsto 1$ is in every member of the family. Second, if f is in the supremum implies that f^* is in the supremum, assume that $f \in \bigcap C_{\overline{U_p}}$. Then f is constant on each $\overline{U_p}$. Hence so is f^*, which is therefore also in $\bigcap C_{\overline{U_p}}$. If f is in the supremum, there is a sequence of $f_1, f_2, \ldots$ with f_n in some $\bigcap C_{\overline{U_{pn}}}$. Hence also $f^* \in \bigcap C_{\overline{U_{pn}}}$. Since $\lim_{n\to\infty} \|f_n^* - f^*\| = \lim_{n\to\infty} \|f_n - f\| = 0$, we find f^* in the supremum.

Finally, let x and y be distinct points in X; we will show that $f(x) \neq f(y)$ for some f in the supremum by distinguishing two cases. For the first case, suppose $x, y \in P$. Since P is finite, it is closed, as is $P \setminus \{x\}$. Hence $\{x\}$ and $P \setminus \{x\}$ are disjoint closed subsets in X, and since X is compact Hausdorff, there are open subsets U and V containing x and $P \setminus \{x\}$, respectively, with disjoint closures. Because U is an open neighbourhood of x and V is an open neighbourhood of p for each $p \in P \setminus \{x\}$, it follows that $C_{\overline{U}} \cap C_{\overline{V}}$ is in the family. But Urysohn's lemma provides a function $f \in C(X)$ satisfying $f(\overline{U}) = \{0\}$ and $f(\overline{V}) = \{1\}$. Hence f is constant on $\overline{U}$ and on $\overline{V}$, so $f \in C_{\overline{U}} \cap C_{\overline{V}}$. Since $y \in P \setminus \{x\} \subseteq \overline{V}$, we find $f(x) = 0 \neq 1 = f(y)$.

For the second case, suppose $x \notin P$, and proceed similarly. Regardless of whether $y \in P$ or not, $\{x\}$ and $P \cup \{y\}$ are disjoint closed subsets, hence there are open sets U and V containing $\{x\}$ and $P \cup \{y\}$, respectively, with disjoint closures. Since V is an open neighbourhood of p for each $p \in P$, we find that $C_{\overline{V}}$ is in the family. Again Urysohn's lemma provides a function $f \in C(X)$ satisfying $f(\overline{U}) = \{0\}$ and $f(\overline{V}) = \{1\}$, and since f is constant on $\overline{V}$, we find $f \in C_{\overline{V}}$, and again $f(x) \neq f(y)$. □

We can now identify the compact elements of $\mathcal{C}(A)$ as the finite-dimensional ones.

Proposition 2.3. *Let A be a C*-algebra. Then $C \in \mathcal{C}(A)$ is compact if and only if it is finite-dimensional.*

Proof. Suppose C is compact, and write X for its spectrum. Let $x \in X$ and consider

$$\mathcal{D} = \{C_{\overline{U}} \mid U \text{ is an open neighbourhood of } x\}.$$

It follows from Lemma 2.2 that $\mathcal{D}$ is directed and $C(X) = \bigvee \mathcal{D}$. Because C is compact, it must equal some element $C_{\overline{U}}$ of $\mathcal{D}$. Since $C(X)$ separates all points of X, so must $C_{\overline{U}}$. But as each $f \in C_{\overline{U}}$ is constant on $\overline{U}$, this can only happen when $\overline{U}$ is a singleton $\{x\}$. This implies $\{x\} = U$, so $\{x\}$ is open. Since $x \in X$ was arbitrary, X must be discrete. Being compact, it must therefore be finite. Hence C is finite-dimensional.

Conversely, assume that C has a finite dimension n. Then it is generated by a finite set $\{p_1, \ldots, p_n\}$ of projections that is *orthogonal* in the sense that $p_i p_j = 0$ for $i \neq j$. Let

$\mathcal{D} \subseteq \mathcal{C}(A)$ be a directed family satisfying $C \subseteq \bigvee \mathcal{D}$. Since each projection p_i is contained in $\bigvee \mathcal{D}$, using Lemma 1.17 we may approximate $\|p_i - p\| < \frac{1}{2}$ with some $D \in \mathcal{D}$ and some projection $p \in D$. Since projections $p \colon X \to \mathbb{C}$ can only take the value 0 or 1, $p \neq p_i$ implies $\|p_i - p\| = 1$, so we must have $p = p_i$. Hence there are $D_1, \ldots, D_n \in \mathcal{D}$ such that $p_i \in D_i$. Since $\mathcal{D}$ is directed, there must be some $D \in \mathcal{D}$ with $D_1, \ldots, D_n \subseteq D$. So $p_1, \ldots, p_n \in D$, which implies that $C \subseteq D$. We conclude that C is compact. □

This leads to the following characterization of algebraicity of $\mathcal{C}(A)$.

Theorem 2.4. *A C*-algebra A is scattered if and only if $\mathcal{C}(A)$ is algebraic.*

Proof. By Proposition 2.3 and Lemma 1.15, the dcpo $\mathcal{C}(A)$ is algebraic if and only if each $C \in \mathcal{C}(A)$ is approximately finite-dimensional. By Theorem 1.21, this is equivalent with scatteredness of A. □

3. Continuity

In this section we characterize C*-algebras A for which $\mathcal{C}(A)$ is continuous.

Definition 3.1. A dcpo is *continuous* when each element satisfies $C = \bigvee \twoheaddownarrow C$.

We start with two lemmas govern the equivalence relation $\sim_B$ on a compact Hausdorff space X defined by $x \sim_B y$ if and only if $b(x) = b(y)$ for each element b of a C*-subalgebra $B \subseteq C(X)$.

Lemma 3.2. *Let X be a compact Hausdorff space and let $B \subseteq C(X)$ a C*-subalgebra. Each equivalence class $[x]_B$ is a closed subset of X.*

Proof. The proof of Proposition 1.8 shows that the quotient $X/{\sim_B}$ is compact Hausdorff. If q is the quotient map, then $q(x) = [x]_B$, which is closed since q is a closed map, being a continuous function between compact Hausdorff spaces. □

Lemma 3.3. *For a compact Hausdorff space X and C*-subalgebra $B \subseteq C(X)$:*

(1) B is finite dimensional if and only if $[x]_B \subseteq X$ is open for each $x \in X$;
(2) if X is connected, B is the (one-dimensional) subalgebra of all constant functions on X if and only if $[x]_B$ is open for some $x \in X$;
(3) if B is infinite-dimensional, there are $x \in X$ and $p \in [x]_B$ such that $B \nsubseteq C_{\overline{U}}$ on each open neighbourhood $U \subseteq X$ of p. If X is connected, this holds for all $x \in X$.

Proof. Fix X and B.

(1) Let $q \colon X \to X/{\sim_B}$ be the quotient map. By definition of the quotient topology, $V \subseteq X/{\sim_B}$ is open if and only if its preimage $q^{-1}[V]$ is open in X. We can regard $[x]_B$ both as a subset of X and as point in $X/{\sim_B}$. Since $[x]_B = q^{-1}[\{[x]_B\}]$, we find that $\{[x]_B\}$ is open in $X/{\sim_B}$ if and only if $[x]_B$ is open in X. Hence $X/{\sim_B}$ is discrete if and only if $[x]_B$ is open in X for each $x \in X$. Now $X/{\sim_B}$ is compact, being a continuous image of a compact space. It is also Hausdorff by Proposition 1.8. Hence $X/{\sim_B}$ is discrete if and only if it is finite. Thus each $[x]_B$ is open in X if and only if B is finite-dimensional.

(2) An equivalence class $[x]_B$ is always closed in X by Lemma 3.2. Assume that it is also open. By connectedness $X = [x]_B$, so $f(y) = f(x)$ for each $f \in B$ and each $y \in X$. Hence B is the algebra of all constant functions on X, and since this algebra is spanned by the function $x \mapsto 1$, it follows that B is one dimensional.

Conversely, if B is the one-dimensional subalgebra of all constant function on X, then for each $f \in B$ there is some $\lambda \in \mathbb{C}$ such that $f(x) = \lambda$ for each $x \in X$. Hence $f(x) = f(y)$ for each $x, y \in X$, whence for each $x \in X$ we have $[x]_B = X$, which is clearly open.

(3) Assume that B is infinite-dimensional. By (i) there must be some $x \in X$ such that $[x]_B$ is not open. Hence there must be a point $p \in [x]_B$ such that $U \nsubseteq [x]_B$ for each open neighbourhood U of p. If X is connected, (ii) implies that $[x]_B$ is not open for any $x \in X$, so p can be chosen as an element of $[x]_B$ for each $x \in X$. In both cases, we have $U \nsubseteq [x]_B$ for any open neighbourhood U of p, hence there is $q \in U$ such that $q \notin [x]_B$. We have $y \in [x]_B$ if and only if $f(x) = f(y)$ for each $f \in B$. So $p \in [x]_B$, and $q \notin [x]_B$ implies the existence of some $f \in B$ such that $f(p) \neq f(q)$. That is, there is some $f \in B$ such that f is not constant on U, so f is certainly not constant on $\overline{U}$. We conclude that for each open neighbourhood U of p there is an $f \in B$ such that $f \notin C_{\overline{U}}$, so $B \nsubseteq C_{\overline{U}}$ for each open neighbourhood U of p. □

We can now characterize the way-below relation on $\mathcal{C}(A)$ in operator-algebraic terms.

Proposition 3.4. *The following are equivalent for a C*-algebra A and $B, C \in \mathcal{C}(A)$:*

(1) $B \ll C$;
(2) $B \in \mathcal{K}(\mathcal{C})$ and $B \subseteq C$;
(3) B is finite-dimensional and $B \subseteq C$.

Proof. By Proposition 2.3, B is finite-dimensional if and only if B is compact, which proves the equivalence between (2) and (3). It is almost trivial that (2) implies (1) by unfolding definitions. For (1) $\Rightarrow$ (3), assume $B \subseteq C$ but B infinite-dimensional. Without loss of generality we may assume that $C = C(X)$ for the spectrum X of C. Lemma 3.3 gives $p \in X$ with $B \nsubseteq C_{\overline{U}}$ for each open neighbourhood $U \subseteq X$ of p. Consider the family

$$\{C_{\overline{U}} \mid U \text{ open neighbourhood of } p\}.$$

By Lemma 2.2, this is a directed family in $\mathcal{C}(A)$ with supremum $C(X)$. However, B is not contained in any member of the family, and so cannot be way below $C = C(X)$. □

This leads to the following characterization of continuity of $\mathcal{C}(A)$.

Theorem 3.5. *A C*-algebra A is scattered if and only if $\mathcal{C}(A)$ is continuous.*

Proof. Let $C \in \mathcal{C}(A)$. It follows from Proposition 3.4 that $\twoheaddownarrow C = \mathcal{K}(\mathcal{C}) \cap \downarrow C$, whence $C = \bigvee \mathcal{K}(\mathcal{C}) \cap \downarrow C$ if and only if $C = \bigvee \twoheaddownarrow C$. Thus continuity and algebraicity coincide. The statement now follows from 2.4. □

4. Meet-continuity

In this section we characterize C*-algebras A for which $\mathcal{C}(A)$ is meet-continuous.

Definition 4.1. A dcpo is *meet-continuous* when it is a meet-semilattice, and

$$C \wedge \bigvee D_i = \bigvee C \wedge D_i \tag{4.1}$$

for each element C and directed subset $\{D_i\}$.

A *closed equivalence relation* on a topological space X is a reflexive, symmetric, and transitive relation R on X that is closed as a subset of $X \times X$ in the product topology. Closed equivalence relations on X form a complete lattice under reverse inclusion. The infimum is simply given by intersection. The supremum is harder to describe, and can in general only be given as

$$\bigvee R_n = \bigcap \{S \subseteq X^2 \text{ closed equivalence relation} \mid \bigcup R_n \subseteq S\}.$$

In the special case where $R \circ S = S \circ R$ we have $R \vee S = R \circ S$ [EE13, Section 6]. By the following lemma, meet-continuity of $\mathcal{C}(C(X))$ comes down to the question whether $R \vee \bigcap S_n \supseteq \bigcap R \vee S_n$ for closed equivalence relations R and $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ on X.

Lemma 4.2. *There is a dual equivalence between closed equivalence relations on a compact Hausdorff space X and $\mathcal{C}(C(X))$, that sends R to*

$$C_R = \{f \in C(X) \mid \forall x, y \in X \colon (x,y) \in R \Rightarrow f(x) = f(y)\}.$$

Proof. The map $R \mapsto C_R$ is essentially bijective because any $C \in \mathcal{C}(C(X))$ corresponds to a quotient $X \twoheadrightarrow \mathrm{Spec}(C)$ of compact Hausdorff spaces, which in turn corresponds to a closed equivalence relation $\sim$ on X by $\mathrm{Spec}(C) = X/\!\sim$. Clearly $R \subseteq S$ if and only if $C_R \supseteq C_S$. □

Notice that C_K, for a C*-algebra $C = C(X)$ and closed subset $K \subseteq X$, is a special case of C_R for the closed equivalence relation $R = \{(x,x) \mid x \in X\} \cup K^2 \subseteq X^2$. In general $C_R = \bigcap_{x \in X} C_{[x]_R}$. It will always be clear from the context which of the two is meant.

Proposition 4.3. *There are closed equivalence relations R and $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ on $[0,1]$ such that $R \vee \bigcap S_n \neq \bigcap R \vee S_n$.*

Proof. We will construct R and S_n similar to the Cantor set; R by keeping the closed middle thirds, and S_n by keeping the closed first and last thirds. Inductively define numbers $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in [0,1]$ indexed by finite strings σ of zeroes and ones:

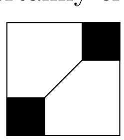
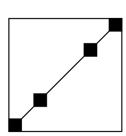
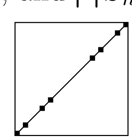
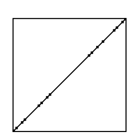
$$\begin{aligned}
a_\epsilon &= 0, & b_\epsilon &= 1/3, & c_\epsilon &= 2/3, & d_\epsilon &= 1, \\
a_{\sigma 0} &= a_\sigma, & b_{\sigma 0} &= a_\sigma + \tfrac{1}{3}(b_\sigma - a_\sigma), & c_{\sigma 0} &= b_\sigma - \tfrac{1}{3}(b_\sigma - a_\sigma), & d_{\sigma 0} &= b_\sigma, \\
a_{\sigma 1} &= c_\sigma, & b_{\sigma 1} &= c_\sigma + \tfrac{1}{3}(d_\sigma - c_\sigma), & c_{\sigma 1} &= d_\sigma - \tfrac{1}{3}(d_\sigma - c_\sigma), & d_{\sigma 1} &= d_\sigma;
\end{aligned}$$

write $\Delta = \{(x,x) \mid x \in [0,1]\}$ for the diagonal, and define

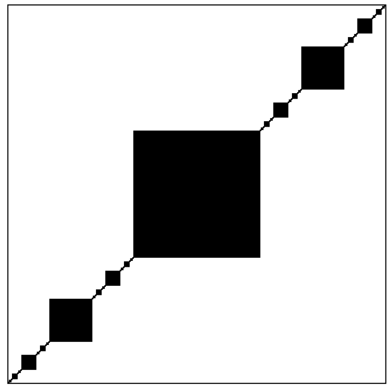
$$R = \Delta \cup \bigcup_{\sigma \in \{0,1\}^*} [b_\sigma, c_\sigma]^2,$$

$$S_n = \Delta \cup \bigcup_{\sigma \in \{0,1\}^n} [a_\sigma, d_\sigma]^2.$$

The S_n are certainly closed equivalence relations, and $\bigcap S_n = \Delta$.

$S_1 =$  $\supseteq S_2 =$  $\supseteq S_3 =$  $\supseteq S_4 =$  $\supseteq \cdots$

We can draw R as follows.

$R =$ 

Clearly R is reflexive and symmetric. It is also transitive: if (x, y) and (y, z) in R have $x \neq y \neq z$, then $(x, y) \in [b_\sigma, c_\sigma]^2$ and $(y, z) \in [b_\tau, c_\tau]^2$ for some $\sigma, \tau \in \{0, 1\}^*$; but if $y \in [b_\sigma, c_\sigma] \cap [b_\tau, c_\tau]$ then $\sigma = \tau$, so $(x, z) \in R$. The set $R \subseteq [0, 1]^2$ is also closed: if $(x_n, y_n) \in R$ is a sequence that converges in $[0, 1]^2$, then either it eventually stays in one block $[b_\sigma, c_\sigma]^2$, or it converges to a point on the diagonal.

In total we see that $R \vee \bigcap S_n = R \vee \Delta = R$. But we now prove that $R \vee S_n = [0, 1]^2$ for any n, so $\bigcap R \vee S_n = [0, 1]^2$, and hence $R \vee \bigcap S_n \neq \bigcap R \vee S_n$. By induction it suffices to show $R \vee S_1 = [0, 1]^2$ and $S_n \subseteq R \vee S_{n+1}$. For the latter it suffices to show $(a_\sigma, d_\sigma) \in R \vee S_{n+1}$ for $\sigma \in \{0, 1\}^n$, which follows from transitivity:

$$a_\sigma = a_{\sigma 0} \, S_{n+1} \, b_{\sigma 0} \, R \, c_{\sigma 0} \, S_{n+1} \, d_{\sigma 0} = b_\sigma \, R \, c_\sigma = a_{\sigma 1} \, S_{n+1} \, b_{\sigma 1} \, R \, c_{\sigma 1} \, S_{n+1} \, d_{\sigma 1} = d_\sigma.$$

Similarly $R \vee S_1 = [0, 1]^2$. □

Theorem 4.4. *A C*-algebra A is scattered if and only if $\mathcal{C}(A)$ is meet-continuous.*

Proof. If A is not scattered, there is an element of $\mathcal{C}(A)$ that is $*$-isomorphic to $C([0, 1])$ by Theorem 1.21. Therefore we may assume without loss of generality that $A = C([0, 1])$. But it now follows from Lemma 4.2 and Proposition 4.3 that $\mathcal{C}(A)$ is not meet-continuous.

If A is scattered, then $\mathcal{C}(A)$ is continuous by Theorem 3.5. But $\mathcal{C}(A)$ is also a complete semilattice, because the intersection $\bigcap C_i$ of a family of commutative C*-subalgebras C_i of A is again a commutative C*-subalgebra. And continuous dcpos that are also semilattices are meet-continuous [GHK+03, I-1.8]. □

We can give another characterization of meet-continuity of $\mathcal{C}(A)$, namely in order-theoretic terms of injective maps $B \to A$.

Theorem 4.5. *The dcpo $\mathcal{C}(A)$ of a C*-algebra A is meet-continuous if and only if $\mathcal{C}(f)$ is Scott continuous for any injective $*$-homomorphism $f \colon B \to A$.*

Proof. Suppose $\mathcal{C}(f)$ is Scott continuous for $*$-homomorphisms $f \colon B \to A$. Let $\{D_i\}$ be a directed family in $\mathcal{C}(A)$, and let $C \in \mathcal{C}(A)$. Write $f \colon C \to A$ for the inclusion. Then $\mathcal{C}(f)$ has an upper adjoint $\mathcal{C}(f)_* \colon \mathcal{C}(B) \to \mathcal{C}(A)$ given by $\mathcal{C}(f)_*(D) = f^{-1}[D]$ [Lin15, Proposition 2], and so $\mathcal{C}(f)_*(D_i) = f^{-1}[D_i] = C \cap D_i$. Because $\mathcal{C}(f)_*$ is Scott continuous,

$$C \cap \bigvee D_i = \mathcal{C}(f)_*\Big(\bigvee D_i\Big) = \bigvee \mathcal{C}(f)_*[D_i] = \bigvee C \cap D_i.$$

Hence $\mathcal{C}(A)$ is meet-continuous.

Now assume $\mathcal{C}(A)$ is meet-continuous and let $f\colon B \to A$ be an injective $*$-homomorphism. Write $\mathcal{C}(f)_*\colon \mathcal{C}(B) \to \mathcal{C}(A)$ for the upper adjoint, and let D_i be a directed family in $\mathcal{C}(B)$. Then $\mathcal{C}(\varphi)_*(D_i) \subseteq \mathcal{C}(f)_*\big(\bigvee D_i\big)$, and hence

$$\bigvee f^{-1}[D_i] \;=\; \bigvee \mathcal{C}(f)_*(D_i) \;\subseteq\; \mathcal{C}(f)_*\big(\bigvee D_i\big) \;=\; f^{-1}\Big[\overline{\bigcup D_i}\Big] \;=\; f^{-1}\Big[\bigvee D_i\Big].$$

If $x \in f^{-1}[\bigvee D_i]$ is self-adjoint, then $f(x) \in \bigvee D_i$, and the smallest C*-algebra $C = C^*(x)$ containing x is in $\mathcal{C}(B)$. Hence $f[C^*(x)] = \mathcal{C}(f)[C^*(x)] = C^*(f(x))$. Also $C^*(f(x)) \subseteq \bigvee D_i$, so that $f[C] \subseteq \bigvee D_i$. Meet-continuity of $\mathcal{C}(A)$ now shows $f[C] = \bigvee f[C] \cap D_i$. Being a C*-subalgebra, $f[B]$ is closed in A, so that the injection f restricts to a $*$-isomorphism and hence a homeomorphism $B \to f[B]$. Observe that $f^{-1}[\overline{S}] = \overline{f^{-1}[S]}$ for $S \subseteq f[B]$. Hence

$$\begin{aligned} C = f^{-1}[f[C]] = f^{-1}\big[\bigvee f[C] \cap D_i\big] &= f^{-1}\big[\overline{\bigcup f[C] \cap D_i}\big] \\ = \overline{f^{-1}\big[\bigcup f[C] \cap D_i\big]} &= \overline{\bigcup f^{-1}[f[C] \cap D_i} \;\subseteq\; \overline{\bigcup f^{-1}[D_i]} = \bigvee f^{-1}[D_i]. \end{aligned}$$

As $x \in C$, it follows that $x \in \bigvee f^{-1}[D_i]$. Finally, an element of $f^{-1}[\bigvee D_i]$ is a linear combination of self-adjoint ones. We conclude that $\mathcal{C}(f)_*\big(\bigvee D_i\big) = \bigvee \mathcal{C}(f)_*(D_i)$. □

5. Atomicity

In this section we characterize the C*-algebras A for which $\mathcal{C}(A)$ is atomistic.

Definition 5.1. Let $\mathcal{C}$ be a partially ordered set with least element 0. An *atom* is an element $C \in \mathcal{C}$ such that $0 < C$, and there is nothing between 0 and C in the sense that $B = C$ whenever $0 < B \leq C$. A partially ordered set is called *atomistic* if each element is the least upper bound of some collection of atoms.

We begin by identifying the atoms in $\mathcal{C}(A)$. Write $C^*(S) \subseteq A$ for the smallest C*-subalgebra containing $S \subseteq A$, and say that $C^*(S)$ is *generated* by S. For example, $C^*(\{p\})$ is just the linear span $\mathrm{Span}\{p, 1-p\}$ for projections $p^2 = p^* = p \in A$; this is two-dimensional unless p is *trivial*, *i.e.* 0 or 1, in which case it collapses to the one-dimensional least element $\mathbb{C}1_A$ of $\mathcal{C}(A)$.

Lemma 5.2. [Ham11, 3.1]. *Let A be a C*-algebra. Then C is an atom in $\mathcal{C}(A)$ if and only if it is generated by a nontrivial projection.*

Proof. Clearly two-dimensional C are atoms in $\mathcal{C}(A)$. Conversely, assume that C is an atom of $\mathcal{C}(A)$. By Theorem 1.7, $C \simeq C(X)$ for a compact Hausdorff space X. If X contains three distinct point x, y, z, then $C(X)$ contains a proper subalgebra $\{f \in C(X) \mid f(x) = f(y)\}$ with dimensional at least two, which contradicts atomicity of C. Hence X must contain exactly two points x and y. Using the *-isomorphism between C and $C(X)$, let $p \in C$ be the element corresponding to the element of $C(X)$ given by $x \mapsto 1$ and $y \mapsto 0$ for $y \neq x$. It follows that $C = \mathrm{Span}\{p, 1-p\}$. □

To characterize atomicity we will need two auxiliary results. The first deals with least upper bounds of subalgebras in terms of generators.

Lemma 5.3. *Let A be a C*-algebra and $C \in \mathcal{C}(A)$. If $\{S_i\}_{i \in I}$ is a family of subsets of C, then each $C^*(S_i)$ is in $\mathcal{C}(A)$, and $C^*\left(\bigcup_{i\in I} S_i\right) = \bigvee_{i \in I} C^*(S_i)$.*

Proof. For any $i \in I$, clearly $C^*(S_i)$ is a commutative C*-subalgebra of A, and hence an element of $\mathcal{C}(A)$.

Writing $S = \bigcup_{i\in I} S_i$, we have $S_j \subseteq C^*(S)$, and so $C^*(S_j) \subseteq C^*(C^*(S)) = C^*(S)$. Therefore, $\bigvee_{i\in I} C^*(S_i)$ is contained in $C^*(S)$. For the inclusion in the other direction, notice that clearly $S \subseteq \bigvee_{i \in I} C^*(S_i)$, whence

$$C^*(S) \subseteq C^*\left(\bigvee_{i\in I} C^*(S_i)\right) = \bigvee_{i\in I} C^*(S_i).$$

This finishes the proof. □

The second auxiliary result deals with subalgebras generated by projections. It shows that projections are the building blocks for C*-algebras A whose dcpos $\mathcal{C}(A)$ are atomistic. This explains why mere approximate finite-dimensionality is not good enough to characterize algebraicity and/or continuity. See also Section 9 below.

Proposition 5.4. *For a C*-algebra A, a C*-subalgebra C is the least upper bound of a collection of atoms of $\mathcal{C}(A)$ if and only if it is generated by projections.*

Proof. Let $C = C^*(P)$, where $P \subseteq A$ is a collection of projections. Then $P \subseteq C$, so that $C^*(\{p\}) \subseteq C$ for each $p \in P$. Since $P = \bigcup_{p\in P}\{p\}$, it follows from Lemma 5.3 that

$$C = C^*(P) = \bigvee_{p\in P} C^*(\{p\}).$$

It now follows from Lemma 5.2 that C is the least upper bound of atoms in $\mathcal{C}(A)$.

Conversely, if $C = \bigvee D_i$ for a collection of atoms D_i in $\mathcal{C}(A)$, we must have $D_i = C^*(\{p_i\})$ for some projections $p_i \in C$. Hence

$$C = \bigvee_{p\in P} C^*(\{p\}) = C^*\Big(\bigcup_{p\in P}\{p\}\Big) = C^*(P),$$

where the second equality used Lemma 5.3. Thus C is generated by projections. □

This leads to the following characterization of atomicity of $\mathcal{C}(A)$.

Theorem 5.5. *A C*-algebra A is scattered if and only if $\mathcal{C}(A)$ is atomistic.*

Proof. By Theorem 2.4 it suffices to prove that $\mathcal{C}(A)$ is algebraic if and only if it is atomistic. Assume that $\mathcal{C}(A)$ is algebraic and let $C \in \mathcal{C}(A)$. If $C = \mathbb{C}1_A$, then C is the least upper bound of the empty set, which is a subset of the set of atoms. Otherwise, it follows from Proposition 2.3 that C is the least upper bound of all its finite-dimensional C*-subalgebras. Since every finite-dimensional C*-algebra is generated by a finite set of projections, it follows from Proposition 5.4 that each element $D \in K(\mathcal{C}(A)) \cap \downarrow C$ can be written as the least upper bound of atoms in $\mathcal{C}(A)$. Hence C is a least upper bound of atoms, so $\mathcal{C}(A)$ is atomistic.

Conversely, assume $\mathcal{C}(A)$ is atomistic and let $C \in \mathcal{C}(A)$. Because C being finite-dimensional implies that it is a least upper bound of $K(\mathcal{C}(A)) \cap \downarrow C$, we may assume C

infinite-dimensional. By Lemma 5.2, $C = \bigvee_{p \in P} C^*(\{p\})$ for some collection P of projections in A. As we must have $P \subseteq C$, all projections in P commute. We may replace P by the set of all projections of C, which we will denote by P as well; then we still have $C = \bigvee_{p \in P} C^*(\{p\})$. Write $\mathcal{F}$ for the collection of all finite subsets of P, and consider the family $\{C^*(F) \in \mathcal{C}(A) \mid F \in \mathcal{F}\}$. If $F \in \mathcal{F}$, then $C^*(F)$ is finite-dimensional, and since finite-dimensional C*-algebras are generated by a finite number of projections, it follows that this family equals $K(\mathcal{C}(A)) \cap \downarrow C$. Now let $F_1, F_2 \in \mathcal{F}$. By Lemma 5.3, $C^*(F_1) \vee C^*(F_2) = C^*(F_1 \cup F_2)$, making the family directed. Then:

$$C = \bigvee_{p \in P} C^*(\{p\}) = \bigvee_{F \in \mathcal{F}} \bigvee_{p \in F} C^*(\{p\}) = \bigvee_{F \in \mathcal{F}} C^*(F),$$

where the third equality used Lemma 5.3. Hence $\mathcal{C}(A)$ is algebraic. □

6. Quasi-continuity and quasi-algebraicity

In this section we show that for dcpos $\mathcal{C}(A)$ of C*-algebras A, the notions of quasi-continuity and quasi-algebraicity, which are generally weaker than continuity and algebraicity, are in fact equally strong.

To define quasi-continuity and quasi-algebraicity we generalize the way below relation of a dcpo $\mathcal{C}$ to nonempty subsets: write $\mathcal{G} \leq \mathcal{H}$ when $\uparrow \mathcal{H} \subseteq \uparrow \mathcal{G}$. This is a pre-order, and we can talk about *directed* families of nonempty subsets. A nonempty subset $\mathcal{G}$ is *way below* another one $\mathcal{H}$, written $\mathcal{G} \ll \mathcal{H}$, when $\bigvee \mathcal{D} \in \uparrow \mathcal{H}$ implies $D \in \uparrow \mathcal{G}$ for some $D \in \mathcal{D}$. Observe that $\{B\} \ll \{C\}$ if and only if $B \ll C$, so we may abbreviate $\mathcal{G} \ll \{C\}$ to $\mathcal{G} \ll C$, and $\{C\} \ll \mathcal{H}$ to $C \ll \mathcal{H}$.

Definition 6.1. For an element C in a dcpo $\mathcal{C}$, define

$$\mathrm{Fin}(C) = \{\mathcal{F} \subseteq \mathcal{C} \mid \mathcal{F} \text{ is finite, nonempty, and } \mathcal{F} \ll C\},$$
$$\mathrm{KFin}(C) = \{\mathcal{F} \in \mathrm{Fin}(C) \mid \mathcal{F} \ll \mathcal{F}\}.$$

The dcpo is *quasi-continuous* if each $\mathrm{Fin}(C)$ is directed, and $C \not\leq D$ implies $D \notin \uparrow \mathcal{F}$ for some $\mathcal{F} \in \mathrm{Fin}(C)$. It is *quasi-algebraic* if each $\mathrm{KFin}(C)$ is directed, and $C \not\leq D$ implies $D \notin \uparrow F$ for some $\mathcal{F} \in \mathrm{KFin}(C)$.

Intuitively, quasi-continuity and quasi-algebraicity relax continuity and algebraicity to allow the information in approximants to be spread out over finitely many observations rather than be concentrated in a single one.

We start by analyzing the way below relation generalized to finite subsets.

Lemma 6.2. *Let A be a C*-algebra, $C \in \mathcal{C}(A)$ and $\mathcal{F} \subseteq \mathcal{C}(A)$. Then $\mathcal{F} \in \mathrm{Fin}(C)$ if and only if $\mathcal{F}$ contains finitely many elements and $F \ll C$ for some $F \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ contain finitely many elements and assume that $F \ll C$ for some $F \in \mathcal{F}$. Let $\mathcal{D}$ be a directed subset of $\mathcal{C}(A)$ such that $C \subseteq \bigvee \mathcal{D}$. Since $F \ll C$, we have $F \subseteq D$ for some $D \in \mathcal{D}$, so $D \in \uparrow \mathcal{F}$. Thus $\mathcal{F} \in \mathrm{Fin}(C)$.

Conversely, suppose $\mathcal{F} \in \mathrm{Fin}(C)$. Then $\mathcal{F} \ll C$ and $\mathcal{F}$ is nonempty and finite. Now $\{C\}$ is a directed subset whose supremum contains C, so there is some $F = \{F_1, \ldots, F_n\} \in \mathcal{F}$ contained in C. Assume for a contradiction that each F_i has infinite dimension. Write X for the spectrum of C, so $C \simeq C(X)$. Lemma 3.3 guarantees the existence of points

$p_1, \dots, p_n \in X$ with $F_j \nsubseteq C_{\overline{U_j}}$ for each open neighbourhood $U_j \subseteq X$ of p_j. In particular, $F_j \nsubseteq \bigcap_{i=1}^n C_{\overline{U_i}}$ for each $i = 1, \dots, n$ and open neighbourhood U_i of p_i. Consider the family

$$\Big\{ \bigcap_{i=1}^n C_{\overline{U_i}} \;\Big|\; U_i \text{ open neighbourhood of } p_i,\ i = 1, \dots, n \Big\}.$$

It is directed and has supremum C by Lemma 2.2. However, no member of the family contains F_i. If $F \in \mathcal{F}$ such that $F \nsubseteq C$, we cannot have $F \subseteq \bigcap C_{\overline{U_i}}$ for any i or neighbourhood U_i of p_i, because the latter is contained in C by construction, contradicting $\mathcal{F} \ll C$. We conclude that there must be a finite-dimensional $F \in \mathcal{F}$ such that $F \subseteq C$. Now $F \ll C$ follows from Proposition 3.4. □

Lemma 6.3. *Let A be a C*-algebra and let $C \in \mathcal{C}(A)$. If $F \ll C$, then $\{F\} \in \mathrm{KFin}(C)$. If $\mathcal{F} \in \mathrm{Fin}(C)$, then $\mathcal{F} \leq \mathcal{F}'$ for some $\mathcal{F}' \in \mathrm{KFin}(C)$.*

Proof. Let $F \ll C$. By Lemma 6.2, we have $\{F\} \in \mathrm{Fin}(C)$. By Lemma 3.4, we have $F \ll F$. Therefore $\{F\} \ll \{F\}$, and so $\{F\} \in \mathrm{KFin}(C)$.

Let $\mathcal{F} \in \mathrm{Fin}(C)$. By Lemma 6.2, there is an $F \in \mathcal{F}$ such that $F \ll C$. The reasoning in the previous paragraph shows $\{F\} \in \mathrm{KFin}\, C$. Since $F \in \mathcal{F}$, we have $F \in \uparrow\mathcal{F}$, and so $\uparrow\{F\} \subseteq \uparrow\mathcal{F}$. We conclude that $\mathcal{F} \leq \mathcal{F}'$ for $\mathcal{F}' = \{F\}$. □

We are now ready to characterize quasi-continuity and quasi-algebraicity of $\mathcal{C}(A)$.

Theorem 6.4. *A C*-algebra A is scattered if and only if $\mathcal{C}(A)$ is quasi-continuous, if and only if $\mathcal{C}(A)$ is quasi-algebraic.*

Proof. If A is scattered, then $\mathcal{C}(A)$ is continuous by Theorem 3.5. Let $C \in \mathcal{C}(A)$, and $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{KFin}(C)$. Since $\mathrm{KFin}(C) \subseteq \mathrm{Fin}(C)$, it follows from Lemma 6.2 that there are $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $F_1, F_2 \ll C$. Hence $F_1, F_2 \in \twoheaddownarrow C$, and since $\twoheaddownarrow C$ is directed by continuity of $\mathcal{C}(A)$, it follows that there is some $F \in \twoheaddownarrow C$ such that $F_1, F_2 \subseteq F$. Setting $\mathcal{F} = \{F\}$, Lemma 6.3 shows $\mathcal{F} \in \mathrm{KFin}(C)$. Because $F_1, F_2 \subseteq F$, we obtain $\mathcal{F} = \{F\} \subseteq \uparrow\mathcal{F}_1 \cap \uparrow\mathcal{F}_2$, making $\mathrm{KFin}(C)$ directed.

Let $B \in \mathcal{C}(A)$ satisfy $C \nsubseteq B$. We have to show $B \notin \uparrow\mathcal{F}$ for some $\mathcal{F} \in \mathrm{KFin}(C)$. Assume for a contradiction that $B \in \uparrow\mathcal{F}$ for each $\mathcal{F} \in \mathrm{KFin}(C)$. Lemma 6.3 shows that $\{F\} \in \mathrm{KFin}(C)$ for each $F \in \twoheaddownarrow C$, whence $F \subseteq B$ for each $F \in \twoheaddownarrow C$. Therefore $\bigvee \twoheaddownarrow C \subseteq B$, and by continuity of $\mathcal{C}(A)$ we have $\bigvee \twoheaddownarrow C = C$, so $C \subseteq B$. This is clearly a contradiction, so $\mathcal{C}(A)$ must be quasi-algebraic.

Now assume that $\mathcal{C}(A)$ is quasi-algebraic and let $C \in \mathcal{C}(A)$. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Fin}(C)$. By Lemma 6.3, there exist elements $\mathcal{F}_1', \mathcal{F}_2' \in \mathrm{KFin}(C)$ such that $\mathcal{F}_i \leq \mathcal{F}_i'$. By quasi-algebraicity, $\mathrm{KFin}(C)$ is directed, so there is an $\mathcal{F} \in \mathrm{KFin}(C)$ such that $\mathcal{F}_1', \mathcal{F}_2' \leq \mathcal{F}$. Hence $\mathcal{F}_1, \mathcal{F}_2 \leq \mathcal{F}$. Since $\mathrm{KFin}(C) \subseteq \mathrm{Fin}(C)$, it then follows that $\mathrm{Fin}(C)$ is directed. Let $B \in \mathcal{C}(A)$ satisfy $C \nsubseteq B$. Assume that $B \in \uparrow\mathcal{F}$ for $\mathcal{F} \in \mathrm{Fin}(C)$. Lemma 6.3 provides $\mathcal{F}' \in \mathrm{KFin}(C)$ with $\mathcal{F} \leq \mathcal{F}'$. But this means that $\uparrow\mathcal{F} \subseteq \uparrow\mathcal{F}'$. Hence $B \in \uparrow\mathcal{F}'$, which contradicts quasi-algebraicity. Therefore we must have $B \notin \uparrow\mathcal{F}$ for each $\mathcal{F} \in \mathrm{Fin}(C)$, making $\mathcal{C}(A)$ quasi-continuous.

Finally, assume $\mathcal{C}(A)$ is quasi-continuous. Let $F_1, F_2 \in \twoheaddownarrow C$. By Lemma 6.2, we have $\{F_1\}, \{F_2\} \in \mathrm{Fin}(C)$, and since $\mathrm{Fin}(C)$ is directed, there is an $\mathcal{F} \in \mathrm{Fin}(C)$ such that $\mathcal{F} \subseteq \uparrow\{F_1\} \cap \uparrow\{F_2\}$. In other words, $F_1, F_2 \subseteq F$ for each $F \in \mathcal{F}$, and since $\mathcal{F} \in \mathrm{Fin}(C)$, Lemma 6.2 guarantees the existence of some F such that $F \ll C$, making $\twoheaddownarrow C$ directed. Let $B = \bigvee \twoheaddownarrow C$. Since $F \subseteq C$ for each $F \in \twoheaddownarrow C$, we have $B \subseteq C$. If $B \neq C$, then $C \nsubseteq B$, so by

quasi-continuity there must be an $\mathcal{F} \in \mathrm{Fin}(C)$ with $B \notin \uparrow\mathcal{F}$. Hence $F \nsubseteq B$ for each $F \in \mathcal{F}$, and in particular, Lemma 6.2 implies the existence of some $F \in \mathcal{F}$ satisfying $F \ll C$, but $F \nsubseteq B$. By definition of B we have $F \subseteq B$ for each $F \ll C$, giving a contradiction. Thus $\mathcal{C}(A)$ is continuous, and by Theorem 3.5, A is scattered. □

7. Scatteredness

In the previous sections, we have seen that the dcpo $\mathcal{C}(A)$ of a C*-algebra A is nice – in the sense of being (quasi-)algebraic, (quasi-)continuous, or atomistic, which are all equivalent – precisely when A is scattered. In this case, we can turn the domain $\mathcal{C}(A)$ itself into the spectrum of another C*-algebra, which this section studies. We will use the Lawson topology, that turns approximation in domains into topological convergence.

Definition 7.1. The *Scott topology* declares subsets $\mathcal{U}$ of a dcpo to be open if $\uparrow\mathcal{U} = \mathcal{U}$, and $D \cap \mathcal{U} \neq \emptyset$ when $\bigvee \mathcal{D} \in \mathcal{U}$. The *Lawson topology* has as basic open subsets $\mathcal{U} \setminus \uparrow\mathcal{F}$ for a Scott open subset $\mathcal{U}$ and a finite subset $\mathcal{F}$.

These topologies capture approximation in the following sense. A monotone function between dcpos is Scott continuous precisely when $\bigvee f[\mathcal{D}] = f\big(\bigvee \mathcal{D}\big)$ for directed subsets $\mathcal{D}$ [GHK+03, II-2.1]. Thus Proposition 1.12 shows that $\mathcal{C}(f)$ is Scott continuous. A monotone function between complete semilattices is Lawson continuous precisely when additionally $\bigwedge f[\mathcal{D}] = f(\bigwedge \mathcal{D})$ for nonempty subsets $\mathcal{D}$ [GHK+03, III-1.8].

Proposition 7.2. *If A is a scattered C*-algebra, then $\mathcal{C}(A)$ is a totally disconnected compact Hausdorff space in the Lawson topology.*

Proof. If A is scattered, then $\mathcal{C}(A)$ is both an algebraic domain and a complete semilattice. Therefore it is compact Hausdorff in the Lawson topology [GHK+03, III-1.11]. Moreover, it follows that $\mathcal{C}(A)$ is zero-dimensional [GHK+03, III-1.14], which for compact Hausdorff spaces is equivalent to being totally disconnected [Wil70, 29.7]. □

It follows from the previous proposition that any scattered C*-algebra A gives rise to another, commutative, C*-algebra $C(X)$ for $X = \mathcal{C}(A)$ with its Lawson topology. Thus we can speak about the domain of commutative C*-subalgebras entirely within the language of C*-algebras.

However, there are two caveats. First, the construction from the previous proposition is not functorial (*i.e.* does not respect unital $*$-homomorphisms), because of rigorous no-go theorems [BH14]. Second, iterating this construction only makes sense in the finite-dimensional case, as the following theorem shows.

Theorem 7.3. *A scattered C*-algebra A is finite-dimensional if and only if $\mathcal{C}(A)$ is scattered in the Lawson topology.*

Proof. Let A have finite dimension, so it is certainly scattered, and $\mathcal{C}(A)$ is algebraic. It follows that a basis for the Scott topology is given by $\uparrow C$ for C compact [GHK+03, II-1.15]. Thus sets of the form $\uparrow C \setminus \uparrow\mathcal{F}$ with C compact and $\mathcal{F}$ finite form a basis for the Lawson topology. Take a nonempty subset $\mathcal{S} \subseteq \mathcal{C}(A)$, and let M be a maximal element of $\mathcal{S}$, which exists by [Lin15, 3.16]. Since M must be finite-dimensional too, it is compact by Proposition 2.3. Hence $\uparrow M$ is Scott open and therefore Lawson open. Maximality of M in $\mathcal{S}$ now gives $\mathcal{S} \cap \uparrow M = \{M\}$, and since $\uparrow M$ is Lawson open, it follows that M is an isolated point of $\mathcal{S}$. Hence $\mathcal{C}(A)$ is scattered.

For the converse, assume A infinite-dimensional. Then $\mathcal{C}(A)$ has a noncompact element C, and $\downarrow C$ contains an isolated point if it intersects some basic Lawson open set in a single point. Hence $\downarrow C \cap \uparrow K \setminus \uparrow \mathcal{F}$ must be a singleton for some finite set $\mathcal{F} \subseteq \mathcal{C}(A)$ and some compact $K \in \mathcal{C}(A)$. In other words, $[K, C] \setminus \mathcal{F}$ is a singleton, where $[K, C]$ is the interval $\{D \in \mathcal{C}(A) \mid K \subseteq D \subseteq C\}$. Since C is infinite-dimensional and scattered (by Theorem 1.21), there are infinitely many atoms in $[K, C]$: for C is atomistic by Theorem 5.5 and hence dominates infinitely many atoms C_i, but K is finite-dimensional by Proposition 2.3, so that $C_i \vee K$, excepting the finitely many $C_i \leq K$, give infinitely many atoms in $[K, C]$. Hence there is no finite subset $\mathcal{F} \subseteq \mathcal{C}(A)$ making $[K, C] \setminus \uparrow \mathcal{F}$ a singleton. We conclude that $\downarrow C$ has no isolated points, so $\mathcal{C}(A)$ cannot be scattered. □

8. Order-scatteredness

Just like there is an established notion of scatteredness for topological spaces and C*-algebras, there is an established notion of scatteredness for partially ordered sets. This section shows that the two notions diverge and should not be confused.

Definition 8.1. A chain is *order-dense* if none of its elements covers another one, *i.e.* if $x < z$ then $x < y < z$ for some z. A poset is *order-scattered* when it does not contain an order-dense chain of at least two points.

Lemma 8.2. *If a C*-algebra A is not scattered, then $\mathcal{C}(A)$ is not order-scattered.*

Proof. If A is not scattered it has a commutative C*-subalgebra with spectrum $[0, 1]$ by Theorem 1.21, so without loss of generality we may assume $A = C([0, 1])$. Consider

$$\{C_{[x,1]} \mid x \in [0, 1)\} \subseteq \mathcal{C}(A).$$

Because $C_{[x,1]} \subseteq C_{[y,1]}$ if and only if $x \leq y$, this set is order-isomorphic to the order-dense chain $[0, 1)$ via the map $C_{[x,1]} \mapsto x$. □

Lemma 8.3. *Let X be an infinite scattered compact Hausdorff space, and let $A = C(X)$. Then $\mathcal{C}(A)$ is not order-scattered.*

Proof. First observe that X must contain an infinite number of isolated points, for if X had only finitely many isolated points $x_1, \ldots, x_n$, then $X \setminus \{x_1, \ldots, x_n\}$ would be closed and hence contain an isolated point x_{n+1}, which is also isolated in X because $X \setminus \{x_1, \ldots, x_n\}$ would be open. Choose a countable set Y of isolated points of X. Note that Y is open, but cannot be closed because X is compact. Let $Z = \overline{Y} \setminus Y$ be the boundary of Y. Since Y is open, Z is closed. Moreover, if $S \subseteq Y$, then $Z \cup S = (\overline{Y} \setminus Y) \cup S = \overline{S} \setminus (Y \setminus S)$ is closed because $Y \setminus S$ consists only of isolated points and hence is open. As Y is countable, we can label its elements by rational numbers $Y = \{x_q\}_{q \in \mathbb{Q}}$. For each $q \in \mathbb{Q}$, set

$$K_q = Z \cup \{x_r \mid r \leq q\},$$

and notice that K_q is closed and infinite. Now $q \mapsto K_q$ is an order embedding of $\mathbb{Q}$ into the set $\mathcal{F}(X)$ of all closed subsets of X with at least two points, partially ordered by inclusion. In turn, $K \mapsto C_K$ is an order embedding of $\mathcal{F}(X)^{\mathrm{op}}$ into $\mathcal{C}(A)$. Composing gives an order embedding $\mathbb{Q}^{\mathrm{op}} \to \mathcal{C}(A)$, and therefore an order-dense chain in $\mathcal{C}(A)$. □

Theorem 8.4. *A C*-algebra A is finite-dimensional if and only if $\mathcal{C}(A)$ is order-scattered.*

Proof. If A is finite-dimensional, then so is each $C \in \mathcal{C}(A)$. Hence all chains in $\mathcal{C}(A)$ have finite length, and therefore cannot be order-dense. That is, $\mathcal{C}(A)$ is order-scattered.

For the converse, assume that A is infinite-dimensional. We distinguish two cases. If A is not scattered, then Lemma 8.2 shows that $\mathcal{C}(A)$ is not order-scattered. If A is scattered, then it has a maximal commutative C*-subalgebra with scattered spectrum X. Because $C(X)$ must be infinite-dimensional [RK91, 4.6.12], X is infinite. Now Lemma 8.3 shows that $\mathcal{C}(C(X))$, and hence $\mathcal{C}(A)$, is not order-scattered. $\square$

9. Projections

An element p of a C*-algebra A is a *projection* when $p^2 = p = p^*$. The projections form a partially ordered set $\mathrm{Proj}(A)$ where $p \leq q$ if and only if $p = pq$. For example, the projections of $C(X)$ are precisely the indicator functions of clopen subsets of X. In general, if p is a projection then so is $1 - p$, and in fact the projections form an orthomodular poset.

Definition 9.1. A partially ordered set P is an *orthomodular poset* when it has a greatest element 1 and comes equipped with an operation $\perp \colon P \to P$ satisfying for all $p, q \in P$:

- $p^{\perp\perp} = p$;
- if $p \leq q$, then $q^\perp \leq p^\perp$;
- p and $p^\perp$ have least upper bound 1;
- if $p \leq q^\perp$, then p and q have a least upper bound;
- if $p \geq q^\perp$ and p and q have greatest lower bound $0 = 1^\perp$, then $x = y^\perp$.

Thus, if a C*-algebra has many projections, it is intuitively rather disconnected. To be precise, a topological space is *Stonean*, or *extremally disconnected*, when the closure of an open set is still open. The operator algebraic version is as follows [Ber72].

Definition 9.2. An *AW*-algebra* is a C*-algebra A such that $\mathrm{Proj}(A)$ is a complete lattice, and every maximal $C \in \mathcal{C}(A)$ is generated by its projections.

Equivalently, but more in line with our purposes, an AW*-algebra is a C*-algebra whose maximal C*-subalgebras have Stonean spectrum [SW15, Theorem 8.2.5]. In particular, any *von Neumann algebra* or *W*-algebra* is an AW*-algebra. In this section we consider variations on $\mathcal{C}(A)$ that cooperate well with projections. We start with approximating an AW*-algebra by its commutative AW*-subalgebras.

Definition 9.3. A AW*-subalgebra of an AW*-algebra A is a C*-subalgebra $C \subseteq A$ that is an AW*-algebra in its own right, with the same suprema of projections as in A. Write $\mathcal{A}(A)$ for the partially ordered set of AW*-subalgebras of A under inclusion.

There is a similar notion $\mathcal{V}(A)$ of W*-subalgebras of a W*-algebra A, that is studied in [DB12]. The about definition is more general, as $\mathcal{V}(A) = \mathcal{A}(A)$ for a W*-algebra A.

Proposition 9.4. *For an AW*-algebra A there is a Galois correspondence*

$$\begin{tikzcd} \mathcal{A}(A) \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \mathcal{C}(A) \arrow[l, shift right] \end{tikzcd}$$

where the upper adjoint maps a C-subalgebra $C \in \mathcal{C}(A)$ to the smallest AW*-subalgebra of A containing it. Hence $\mathcal{A}(A)$ is a dcpo.*

Proof. Write $C' = \bigcap\{W \in \mathcal{A}(A) \mid C \subseteq W\}$ for the smallest W*-subalgebra of A containing $C \in \mathcal{C}(A)$. If $C \subseteq D$, then clearly $C' \subseteq D'$. By construction we have $C' \subseteq W$ if and only if $C \subseteq W$, for $C \in \mathcal{C}(A)$ and $W \in \mathcal{A}(A)$. Finally, notice that $W' = W$ for $W \in \mathcal{A}(A)$. $\square$

Next we characterize the AW*-algebras A whose dcpos $\mathcal{C}(A)$ and $\mathcal{A}(A)$ are continuous, extending [DB12, 6.1]. This needs the following lemma.

Lemma 9.5. *Compact Hausdorff spaces that are scattered and Stonean must be finite.*

Proof. Consider the open and discrete set

$$U = \{x \in X \mid \{x\} \text{ is closed and open}\}.$$

Assume that $X \setminus U \neq \emptyset$. By scatteredness, $\{x\}$ is open in $X \setminus U$ for some $x \in X \setminus U$. Therefore $X \setminus (U \cup \{x\}) = (X \setminus U) \setminus \{x\}$ is closed in $X \setminus U$ and hence closed in X. Thus both $\{x\}$ and $X \setminus (U \cup \{x\})$ are closed subsets. Since X is compact Hausdorff, there are disjoint open subsets V_1 and V_2 containing x and $X \setminus (U \cup \{x\})$, respectively. We may assume V_1 is closed because X is Stonean. Observe that V_1 is infinite; otherwise $V_1 \setminus \{x\}$ would be closed and $\{x\} = V_1 \setminus (V_1 \setminus \{x\})$ open, contradicting $x \notin U$. Hence $V_1 \setminus \{x\}$ is infinite, too. Pick two disjoint infinite subsets W_1, W_2 covering $V_1 \setminus \{x\}$. Since W_i is contained U, it must be open. If W_i were closed, then it is compact, contradicting that it is both discrete (as subset of U) and infinite. So $W_i \subsetneq \overline{W_i} \subseteq V_1$. Moreover, $\overline{W_i} \cap W_j = \emptyset$ for $i \neq j$, and $W_1 \cup W_2 = V_1 \setminus \{x\}$, so $\overline{W_i} = W_i \cup \{x\}$ whence $\overline{W_1} \cap \overline{W_2} = \{x\}$. Since X is Stonean, $\overline{W_i}$ is open, whence $\{x\}$ is open, contradicting $x \notin U$. Hence $X = U$, and since U is discrete and compact, it must be finite. $\square$

Theorem 9.6. *The following are equivalent for an AW*-algebra A:*

- $\mathcal{C}(A)$ *is continuous;*
- $\mathcal{C}(A)$ *is algebraic;*
- $\mathcal{A}(A)$ *is continuous;*
- $\mathcal{A}(A)$ *is algebraic;*
- A *is finite dimensional.*

Proof. If $\mathcal{C}(A)$ is algebraic or continuous, then A is scattered by Theorems 2.4 and 3.5. Theorem 1.21 then implies that all maximal commutative C*-subalgebras of A are scattered. But maximal C*-subalgebras are automatically AW*-algebras by Proposition 9.4, and scattered commutative AW*-algebras are finite-dimensional by Lemma 9.5. Since all maximal commutative C*-subalgebras of A are finite-dimensional, so is A itself [RK83, 4.12].

Clearly, finite-dimensional A have continuous/algebraic dcpos $\mathcal{C}(A)$ and $\mathcal{A}(A)$. Finally, we show that A is finite-dimensional when $\mathcal{A}(A)$ is continuous by contraposition. Suppose A is infinite-dimensional. Pick a maximal commutative C*-subalgebra $C \subseteq A$; its Stonean spectrum X will have infinitely many points, and by compactness hence a non-isolated point x. Any other point $y_1 \in X$ is separated from x by a clopen U_1, and induction gives a sequence of disjoint clopens $U_1, U_2, \ldots$. Their indicator functions form an infinite set P of pairwise orthogonal projections in A.

Let $I \subseteq P$ be an infinite subset with infinite complement. Its supremum $p = \bigvee I$ is nonzero. Choosing some nonzero $q \in P \subseteq I$ gives $rq = 0$ for each $r \in I$, and hence $pq = 0$ [Ber72, Proposition 3.6], so that $p \neq 1$. By Lemma 5.2, $C^*(p)$ is an atom in $\mathcal{C}(A)$. Consider the directed family $\{C^*(F) \mid F \subseteq P \text{ finite}\}$ of elements of $\mathcal{A}(A)$, whose supremum contains p. We will show that no element $C = C^*(\{p_1, \ldots, p_n\})$ of the family can contain p. Observe that $p_{n+1} = 1 - \sum_{i=1}^{n} p_i \in C$ is orthogonal to each $p_1, \ldots, p_n$, and hence $\sum_{i=1}^{n+1} p_i =$

1. Therefore $C = C^*(\{p_1, \ldots, p_{n+1}\}) = \mathrm{Span}\{p_1, \ldots, p_{n+1}\} = \mathrm{Span}\{p_1, \ldots, p_n, 1\}$. If p were in C, we could thus write it as $p = \sum_{i=1}^n \lambda_i p_i + \lambda 1$ for some coefficients λ, λ_i. Pick a nonzero element $q \in I$ distinct from $p_1, \ldots, p_n$. Because $q \leq p$ we find $q = qp = \sum_{i=1}^n \lambda_i q p_i + \lambda q = \lambda q$, whence $\lambda = 1$. Now pick a nonzero element $q \in P \setminus I$ distinct from $p_1, \ldots, p_n$. Then $qp = 0$ and hence $qp_i = 0$ for each $i = 1, \ldots, n$. Thus $q = \sum_{i=1}^n \lambda_i q p_i + q = qp = 0$, and p cannot be contained in C. Therefore $C^*(p)$ is not compact, but since it is an atom of $\mathcal{A}(A)$, it is way above the bottom element only. Hence $\bigvee \twoheaddownarrow C^*(p) \neq C^*(p)$, and $\mathcal{A}(A)$ is not continuous. □

We conclude that, at least from a domain-theoretic perspective of approximating quantum computations by classical ones, von Neumann algebras are a lot less interesting than C*-algebras. Any C*-algebra A can be turned into a von Neumann algebra by taking its double dual A^{**}, also called the *enveloping von Neumann algebra* [Tak00]. This in fact gives an adjunction of categories showing that von Neumann algebras form a reflexive subcategory of C*-algebras [Dau72, 3.2]. There are many examples of C*-algebras A for which $\mathcal{C}(A)$ is continuous but $\mathcal{C}(A^{**})$ is not: any infinite-dimensional scattered C*-algebra will do, such as $C(X)$ for the infinite compact Hausdorff scattered spaces X of Example 1.19.

Finally, we consider the variation $\mathcal{C}_{\mathrm{AF}}(A)$ of the partially ordered set of approximately finite-dimensional commutative C*-subalgebras of A, and show that it is isomorphic to all Boolean subalgebras of projections.

Definition 9.7. Let P be an orthomodular poset. A subset B that is closed under $\perp$ and for which the meet and join of any two elements in B exists in P and is contained in B is called a *Boolean subalgebra* of P. We denote the set of all Boolean subalgebras of P by $\mathcal{B}(P)$, which we partially order by inclusion.

Theorem 9.8. *If A is a C*-algebra, then $\mathcal{C}_{\mathrm{AF}}(A) \simeq \mathcal{B}(\mathrm{Proj}(A))$ via $C \mapsto \mathrm{Proj}(C)$.*

Proof. Let A be a C*-algebra. Write $f \colon \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{B}(\mathrm{Proj}(A))$ for the map $C \mapsto \mathrm{Proj}(C)$. Define $g \colon \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{C}_{\mathrm{AF}}(A)$ by $g(B) = C^*(B)$; the proof of Proposition 5.4 shows that this is well-defined. Both f and g are clearly monotone. Moreover, if $C \in \mathcal{C}_A F(A)$, then $C^*(\mathrm{Proj}(C)) = C$, so that $g(f(C)) = C$. Now let $B \in \mathcal{B}(\mathrm{Proj}(A))$. Say that B is isomorphic to the Boolean algebra $B(X)$ of clopen subsets of the Stone space X. There is an isomorphism $B(X) \simeq \mathrm{Proj}(C(X))$. Hence we may assume that $B = \mathrm{Proj}(C(X))$ for some Stone space X. Now $C^*(B) = C^*(\mathrm{Proj}(C(X))) = C(X)$, whence $\mathrm{Proj}(C^*(B)) = B$, so that $f(g(B)) = B$. Therefore f and g are inverses. □

It follows that $\mathcal{C}_{\mathrm{AF}}(A)$ is an algebraic complete semilattice, whose compact elements are precisely the finite-dimensional commutative C*-subalgebras [Lin16, 6.1.2]. This generalizes [GKM72] from Boolean algebras to arbitrary orthomodular posets. See also [Heu14c].

Remark 9.9. Because an approximately finite-dimensional C*-algebra is a directed colimit of finite-dimensional C*-algebras, one might wonder whether the functors $\mathcal{C}$ or $\mathcal{C}_{\mathrm{AF}}$ preserve directed colimits. The category of dcpos (and Scott-continuous functions) has all such directed colimits [GHK$^+$03, Proposition IV-4.3]. When all the maps in the diagram have Scott continuous upper adjoints, the limit of these adjoints equals the colimit of the diagram [GHK$^+$03, Theorem IV.4-5]. Now, if a C*-algebra A is the dense union of a directed family of finite-dimensional C*-algebras A_i, then $\mathcal{C}(A_i)$ is a directed diagram of algebraic dcpos. But then $\mathrm{colim}\, \mathcal{C}(A_i)$ is also an algebraic dcpo [GHK$^+$03, Corollary IV-4.11]. Hence $A \mapsto \mathcal{C}(A)$ can at most preserve directed colimits for scattered A, and the functor $\mathcal{C}$ cannot preserve directed colimits of arbitrary finite-dimensional C*-algebras.

Nor can the functor $\mathcal{C}_{\mathrm{AF}}$ preserve directed colimits of scattered approximately finite-dimensional C*-algebras $A = \operatorname{colim} A_i$. To see this, note that $\mathcal{C}_{\mathrm{AF}}(A_i) = \mathcal{C}(A_i)$ for each finite-dimensional A_i only contains compact elements by Proposition 2.3. Therefore all elements of $\operatorname{colim} \mathcal{C}_{\mathrm{AF}}(A)$ are compact, too. But $\mathcal{C}_{\mathrm{AF}}(A)$ must contain non-compact elements if A is infinite-dimensional. To see this, not that an infinite-dimensional but approximately finite-dimensional C*-algebra A has a strictly increasing sequence of finite-dimensional C*-subalgebras A_i. Choose maximal commutative C*-subalgebras $C_i \subseteq A_i$ with $C_1 \subsetneq C_2 \subsetneq C_3 \subsetneq \cdots$. Then $\bigvee_{i=1}^{n} C_i$ is an infinite-dimensional but approximately finite-dimensional commutative C*-subalgebra of A.

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